GPDs and TMDs

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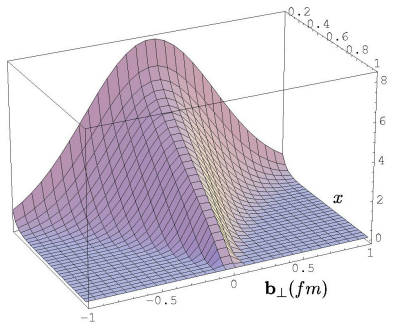
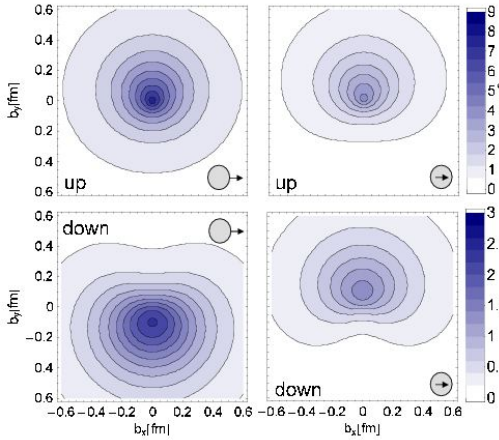
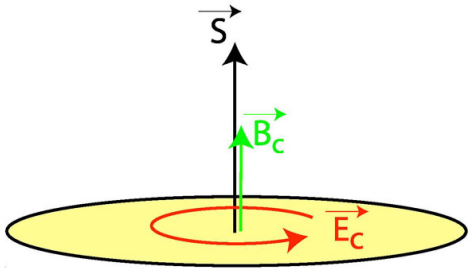
Outline

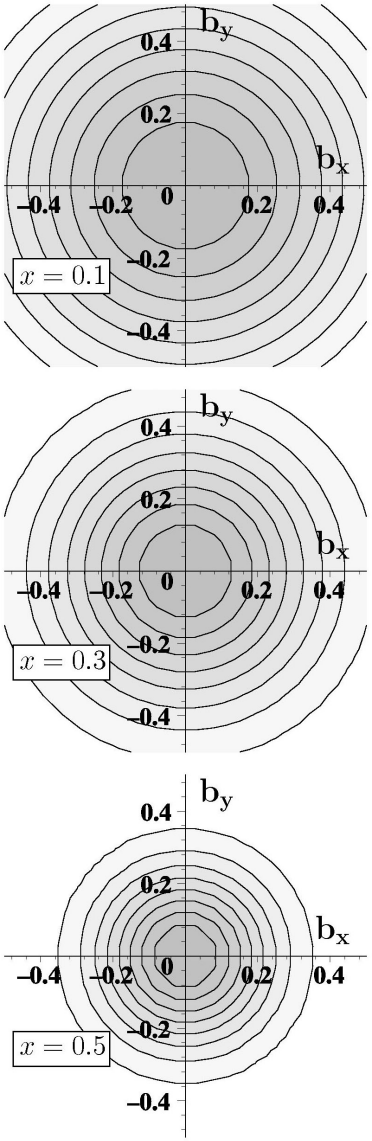
- GPDs: Motivation
  - impact parameter dependent PDFs
  - ↪ Ji relation
- DVCS $\overset{Q^2\ evol.}{\rightsquigarrow}$ GPDs
- ↪ Ji relation (poor man's derivation)
- comparison Jaffe↔Ji decomposition
- ↪ torque in DIS
- Summary

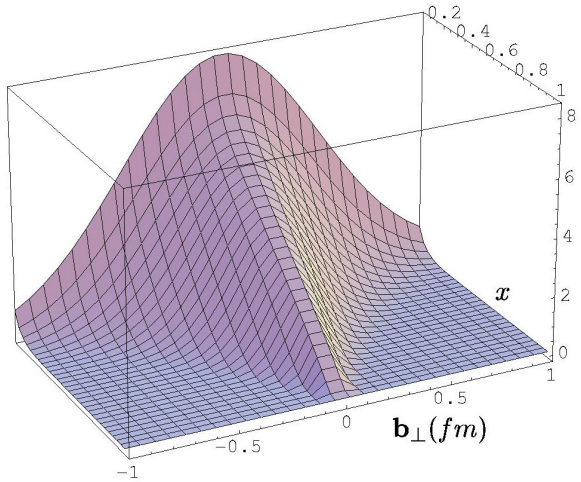


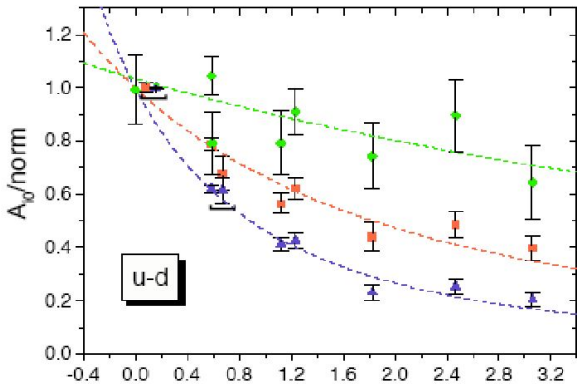




$q(x, \mathbf{b}_\perp)$ for unpol. p



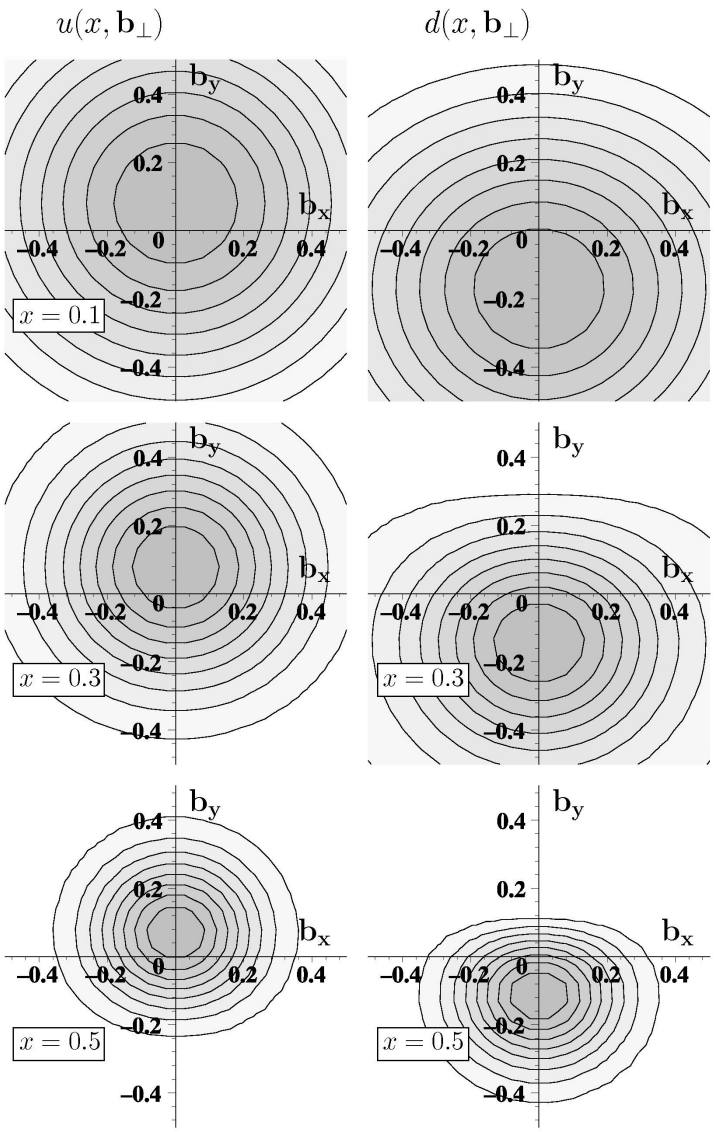


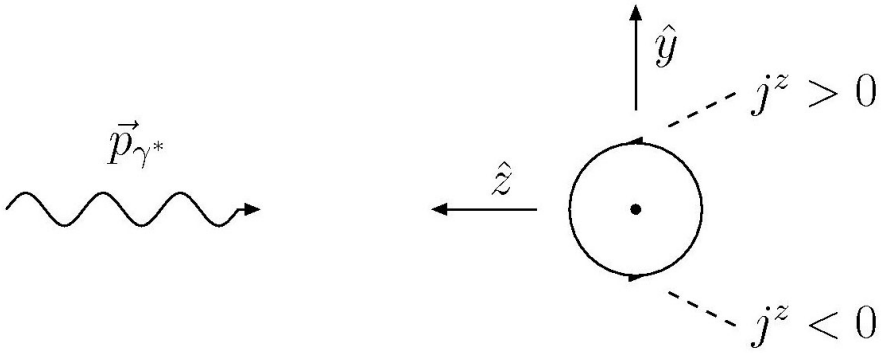


unpolarized proton

- $q(x, \mathbf{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} H(x, 0, -\mathbf{\Delta}_\perp^2) e^{-i\mathbf{b}_\perp \cdot \mathbf{\Delta}_\perp}$
- x = momentum fraction of the quark
- $\vec{b} = \perp$ distance of quark from $\perp$ center of momentum
- small x: large 'meson cloud'
- larger x: compact 'valence core'
- $x \to 1$: active quark becomes center of momentum

$\hookrightarrow$ $\vec{b}_\perp \to 0$ (narrow distribution) for $x \to 1$





proton 'polarized in $+\hat{x}$ direction'

no axial symmetry!

$$q(x, \mathbf{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} H_q(x, 0, -\Delta_\perp^2) e^{-i\mathbf{b}_\perp \cdot \Delta_\perp}$$

$$-\frac{1}{2M}\frac{\partial}{\partial b_y} \int \frac{d^2\Delta_\perp}{(2\pi)^2} E_q(x, 0, -\Delta_\perp^2) e^{-i\mathbf{b}_\perp \cdot \Delta_\perp}$$

Physics: relevant density in DIS is $j^+ \equiv j^0 + j^3$ and left-right asymmetry from j^3

Impact parameter dependent quark distributions

proton 'polarized in $+\hat{x}$ direction' & localized in the $\perp$ direction

$$q(x,\mathbf{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} H_q(x,0,-\Delta_\perp^2) e^{-i\mathbf{b}_\perp\cdot\Delta_\perp} - \frac{1}{2M}\frac{\partial}{\partial b_y}\int \frac{d^2\Delta_\perp}{(2\pi)^2} E_q(x,0,-\Delta_\perp^2) e^{-i\mathbf{b}_\perp\cdot\Delta_\perp}$$

spin + relativity = weirdness ($\rightarrow$ Naomi Makins)

above $q(x,\mathbf{b}_\perp)$ calculated in $\perp$ localized state
$|'\hat{x}'\rangle \equiv |p^+, \mathbf{R}_\perp{=}0, +\rangle + |p^+, \mathbf{R}_\perp{=}0, -\rangle$ which is <u>not</u> eigenstate of $\perp$ nucleon spin

- due to presence of $\mathbf{p}_\perp \neq 0$
- $\pm$ refers to light-front helicity states (issue when $\mathbf{p}_\perp \neq 0$)

distribution in delocalized wave packet

MB, PRD72, 094020 (2005)
$q_\psi(x,\mathbf{b}_\perp) = \int d^2r_\perp q(x, b_\perp - r_\perp) \left(|\psi(\mathbf{r}_\perp)|^2 - \frac{1}{2M}\frac{\partial}{\partial r}_\perp |\psi(\mathbf{r}_\perp)|^2\right)$
two contributions to $\perp$ shift

- intrinsic shift relative to center of momentum $\mathbf{R}_\perp$
- overall shift of $\mathbf{R}_\perp$ for $\perp$ polarized nucleon

spherically symmetric wave packet has center of momentum off-center:

- illustrate this relativistic effect using bag model wave functions:

$$\psi = \begin{pmatrix} f(r) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M_N} f(r) \end{pmatrix} \chi \qquad \text{with} \qquad \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  $\int d^3r f^2(r) = 1$, take limit of large 'radius' R for wave packet
- evaluate $T_q^{0z} = \frac{i}{2}\bar{q}\left(\gamma^0\partial^z + \gamma^z\partial^0\right) q$ in this state
- $\psi^\dagger \partial_z \psi$ even under $y \to -y$, i.e. no contribution to $\langle y T_q^{0z} \rangle$
- use $i\psi^\dagger\gamma^0\gamma^z\partial^0\psi = E\psi^\dagger\gamma^0\gamma^z\psi$

$$\begin{aligned} \langle T^{0z} y \rangle &= E \int d^3r \psi^\dagger \gamma^0 \gamma^z \psi y = E \int d^3r \psi^\dagger \begin{pmatrix} 0 & \sigma^z \\ \sigma^z & 0 \end{pmatrix} \psi y \\ &= \frac{2E}{E+M_N} \int d^3r \chi^\dagger \sigma^z \sigma^y \chi f(r)(-i)\partial^y f(r) y \\ &= \frac{E}{E+M_N} \int d^3r f^2(r) \overset{R\to\infty}{\longrightarrow} \frac{1}{2} \end{aligned}$$

$\hookrightarrow$ p pol. in $+\hat{x}$ direction has CoM shifted by $\frac{1}{2M_N}$ in $+\hat{y}$ direction!

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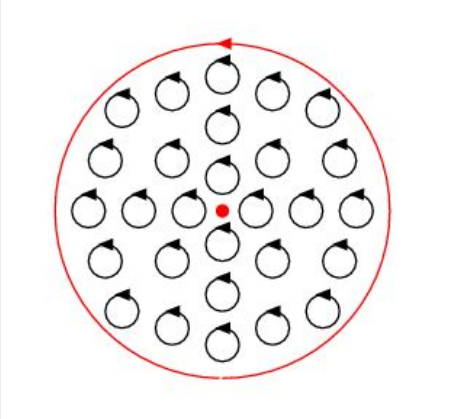
origin of 'shift' of CoM

- nucleon polarization: $\odot$
- counterclockwise momentum density from lower component
- $p \sim \frac{1}{R}$, but $y \sim R$

$\hookrightarrow$ $\langle T^{++} y \rangle = \mathcal{O}(1)$



Total (Spin+Orbital) Quark Angular Momentum

$$J_q^x = L_q^x + S_q^x = \int d^3r \left[y T_q^{0z}(\vec{r}) - z T_q^{0y}(\vec{r}) \right]$$

- $T_q^{\mu\nu}(\vec{r})$ energy momentum tensor $(T_q^{\mu\nu}(\vec{r}) = T_q^{\nu\mu}(\vec{r}))$
- $T_q^{0i}(\vec{r})$ momentum density $[P_q^i = \int d^3r T_q^{0i}(\vec{r})\ ]$
- think: $(\vec{r} \times \vec{p})^x = y p^z - z p^y$

relate to impact parameter dependent quark distributions $q_\psi(x, \mathbf{r}_\perp)$:

Consider spherically symmetric wave packet with nucleon polarized in $+\hat{x}$ direction

- eigenstate under rotations about x-axis

$\hookrightarrow$ both terms in J_q^x equal:
$J_q^x = 2 \int d^3r\, y T_q^{0z}(\vec{r}) = \int d^3r\, y \left[T_q^{0z}(\vec{r}) + T_q^{z0}(\vec{r}) \right]$

- $\int d^3r\, y T_q^{00}(\vec{r}) = 0 = \int d^3r\, y T_q^{zz}(\vec{r})$

$$\Rightarrow \quad J_q^x = \int d^3r\, y T_q^{++}(\vec{r}) \quad \text{with} \quad T^{++} \equiv T^{00} + T^{0z} + T^{z0} + T^{zz}$$

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- $\int dx\, x q(x, \mathbf{r}_\perp) = \frac{1}{2m_N} \int dr^z T^{++}(\vec{r})$
(note: here x is momentum fraction and not r^x)

$\hookrightarrow$ $\langle\psi|J_q^x|\psi\rangle = m_N \int dx \int d^2b_\perp x b^y q_\psi(x, \mathbf{b}_\perp)$

distribution in delocalized wave packet (pol. in $+\hat{x}$ direction)

$q_\psi(x, \mathbf{b}_\perp) = \int d^2 r_\perp q(x, b_\perp - r_\perp) \left(|\psi(\mathbf{r}_\perp)|^2 - \frac{1}{2M} \frac{\partial}{\partial r_y} |\psi(\mathbf{r}_\perp)|^2 \right)$ with

$$q(x,\mathbf{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} H_q(x, 0, -\Delta_\perp^2) e^{-i\mathbf{b}_\perp \cdot \Delta_\perp} - \frac{1}{2M} \frac{\partial}{\partial b_y} \int \frac{d^2\Delta_\perp}{(2\pi)^2} E_q(x, 0, -\Delta_\perp^2) e^{-i\mathbf{b}_\perp \cdot \Delta_\perp}$$

two contributions to $\perp$ shift

- intrinsic shift relative to center of momentum $\mathbf{R}_\perp$
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insert into $\langle \psi | J_q^x | \psi \rangle = \int dx \int d^2 b_\perp q_\psi(x, \mathbf{b}_\perp)$ MB, PRD72, 094020 (2005)

$\langle \psi | J_q^x | \psi \rangle = \frac{1}{2} \int dx\, x \left[H_q(x, 0, 0) + E_q(x, 0, 0) \right]$ (here: derived for $\vec{p} = \vec{0}$ only!)

- X.Ji (1996): rotational invariance $\Rightarrow$ apply to all components of $\vec{J}$
- result for J_q^z also applies to $p_z \neq 0$
- partonic interpretation ($\perp$ shift) exists only for $\perp$ components of $\vec{J}_q$!

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gauge invariance

- matrix element of $T_q^{++} = \bar{q}\gamma^+ i\partial^+ q$ in $A^+=0$ gauge same as that of $\bar{q}\gamma^+\left(i\partial^+ - gA^+\right)q$ in any gauge

$\hookrightarrow$ identify $\frac{1}{2}\int dx\, x\,[H(x,0,0)+E(x,0,0)]$ with J_q in decomposition where $\vec{L}_q = \int d^3x \langle P,S|q^\dagger(\vec{x})\left(\vec{x}\times i\vec{D}\right)q(\vec{x})|P,S\rangle$

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caution!

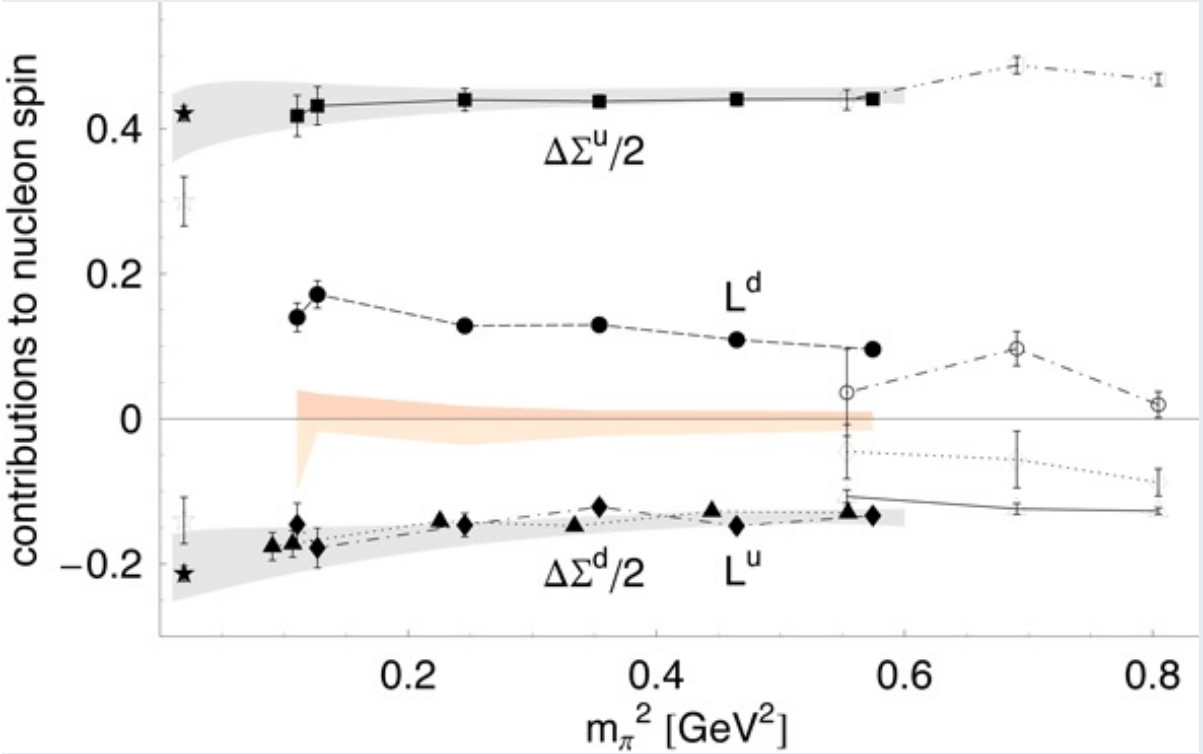
- made heavily use of rotational invariance

$\hookrightarrow$ itentification $\langle\psi|J_q^x|\psi\rangle = \frac{1}{2}\int dx\, x\,[H(x,0,0)+E(x,0,0)]$ does not apply to unintegrated quantities

- $\int d^2\Delta_\perp e^{-i\mathbf{b}_\perp\cdot\mathbf{\Delta}_\perp}\frac{x}{2}\left[H(x,0,-\Delta_\perp^2)+E(x,0,-\Delta_\perp^2)\right]$ not equal to $J^z(\mathbf{b})_\perp$
- $J_q(x) \equiv \frac{x}{2}\left[H_q(x,0,0)+E_q(x,0,-\Delta_\perp^2)\right]$ not x-distribution of angular momentum $J_q^z(x)$ in long. pol. target

regardless whether one takes gauge covariant definition or not

lattice: QCDSF



- no disconnected quark loops
- chiral extrapolation

$$J^q = \frac{1}{2}\int dx\, x\,[H(x,0,0) + E(x,0,0)] \qquad L^q = J^q - \frac{1}{2}\Delta\Sigma^q$$

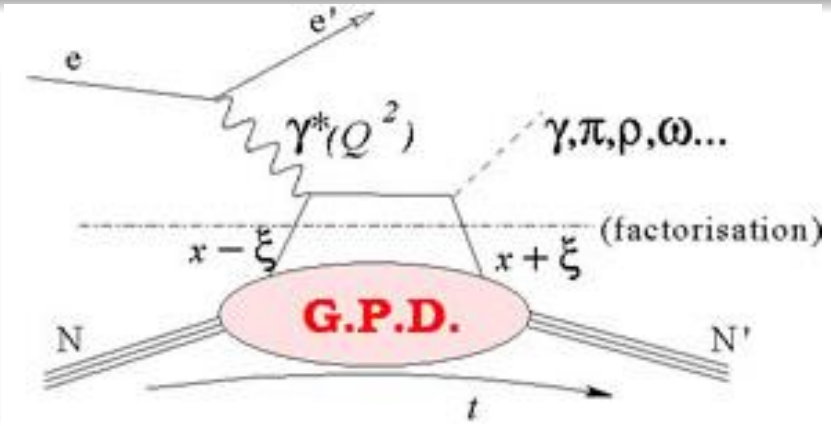
$L^u + L^d \approx 0$

signs of L^q counter-intuitive

$\mathcal{A}_{DVCS} \rightsquigarrow GPDs$

interesting GPD physics:

- $J_q = \int_0^1 dx x \left[H(x,\xi,0) + E(x,\xi,0)\right]$ requires $GPDs(x,\xi,0)$ for (common) fixed ξ for all x
- $\perp$ imagingrequires $GPDs(x,\xi=0,t)$



- ξ longitudinal momentum transfer on the target $\xi = \frac{p^{+\prime} - p^+}{p^{+\prime} + p^+}$
- x (average) momentum fraction of the active quark $x = \frac{k^{+\prime} + p^+}{p^{+\prime} + p^+}$

$\Im\mathcal{A}_{DVCS}(\xi,t) \longrightarrow GPD(\xi,\xi,t)$

- only sensitive to 'diagonal' $x = \xi$
- limited ξ range

$\Re\mathcal{A}_{DVCS}(\xi,t) \longrightarrow \int_{-1}^1 dx \frac{GPD(x,\xi,t)}{x-\xi}$

- limited ξ range
- most sensitive to $x \approx \xi$
- some sensitivity to $x \neq \xi$, but

Polynomiality/Dispersion Relations (GPV/AT DI)

$$\Re\mathcal{A}(\xi,t,Q^2) = \int_{-1}^1 dx \frac{H(x,\xi,t,Q^2)}{x-\xi} = \int_{-1}^1 dx \frac{H(x,x,t,Q^2)}{x-\xi} + \Delta(t,Q^2)$$

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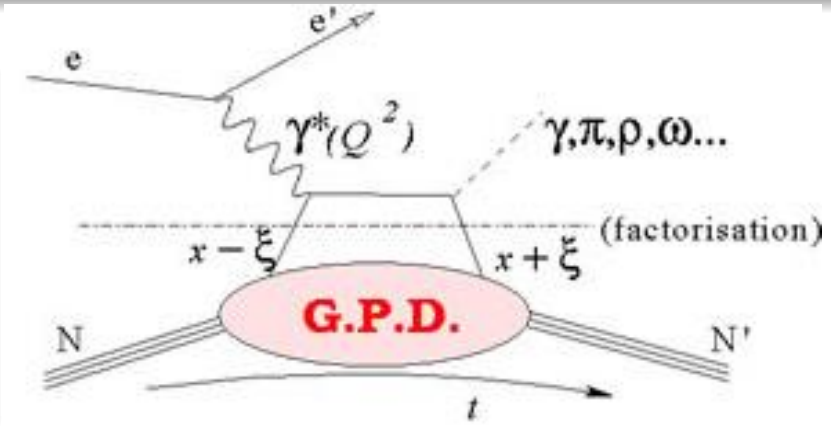

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- Can 'condense' all information contained in contained in $\mathcal{A}_{DVCS}$ (fixed Q^2) into $GPD(x,x,t,Q^2)$ & $\Delta(t,Q^2)$
- if two models both satisfy polynomiality and are equal for $x = \xi$ (but not for $x \neq \xi$) and have same $\Delta(t,Q^2)$ then DVCS at fixed Q^2 cannot distinguish between the two models

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need Evolution!

$$\mu^2 \frac{d}{d\mu^2} H^{q(-)}(x,\xi,t) = \int_{-1}^{1} dx' \frac{1}{|\xi|} V_{\rm NS}\left(\frac{x}{\xi},\frac{x'}{\xi}\right) H^{q(-)}(x',\xi,t)$$

- Q^2 evolution changes x distribution in a known way for fixed ξ

$\hookrightarrow$ measure Q^2 dependence to disentangle x vs. ξ dependence

$$GPD(x,\xi,t,Q^2) = (1-x^2)\sum_{n=0}^{\infty} C_n^{3/2}(x) \sum_{m=0(even)}^{n} a_{nm}(\xi)\mathcal{C}_{n-m}(\xi,t,Q^2)$$

- $C_n^{3/2}(x)$ Gegenbauer polynomials; $a_{nm}(\xi)$ known polynomial
- $\mathcal{C}_k(\xi,t,Q^2)$ unknown, but evolve with known power of $\alpha_s(Q^2)$
- consider $x=\xi$ (relabel: $k=n-m$)

$$GPD(\xi,\xi,t,Q^2) = (1-\xi^2)\sum_{k=0}^{\infty} \mathcal{C}_k(\xi,t,Q^2) f_k(\xi) \qquad (2)$$

with $f_k(\xi) = \sum_{m=0(even)}^{\infty} a_{m+k,m}(\xi) C_{m+k}^{3/2}(\xi)$ known function.

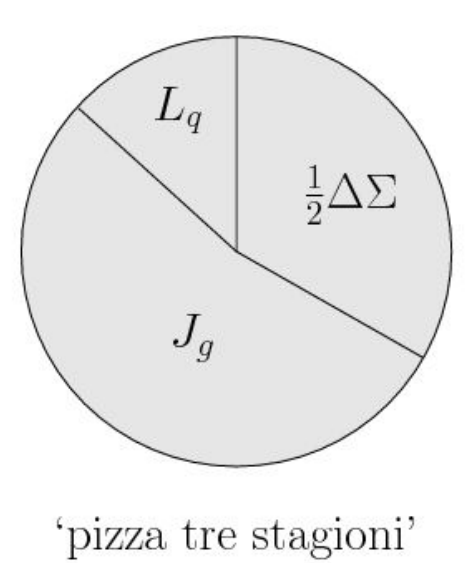
- for fixed ξ, each term in (1) evolves with different γ_k

$\hookrightarrow$ from Q^2-dependence of $GPD(\xi,\xi,t,Q^2)$ (fixed ξ and t) over 'wide' range of Q^2, in principle possible to determine $\mathcal{C}_k(\xi,t,Q^2)$

$\hookrightarrow$ $GPD(x,\xi,t,Q^2)$ for $x \neq \xi$ model-independently!

need EIC as QCD evolution is slow...($\rightarrow$ Aschenauer et al.)

Ji decomposition



$$\frac{1}{2} = \sum_q \frac{1}{2}\Delta q + L_q + J_g$$

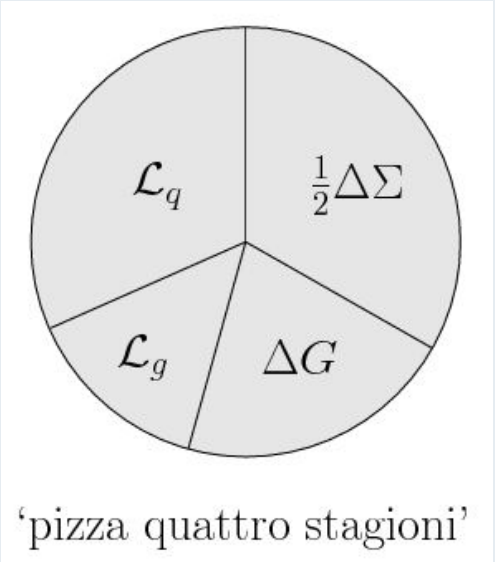
$$\frac{1}{2}\Delta q = \frac{1}{2}\int d^3x \, \langle P,S| \, q^\dagger(\vec{x})\Sigma^3 q(\vec{x}) \, |P,S\rangle$$

$$L_q = \int d^3x \langle P,S| q^\dagger(\vec{x})\left(\vec{x}\times i\vec{D}\right)^3 q(\vec{x}) |P,S\rangle$$

$$J_g = \int d^3x \, \langle P,S| \left[\vec{x}\times\left(\vec{E}\times\vec{B}\right)\right]^3 |P,S\rangle$$

- $i\vec{D} = i\vec{\partial} - g\vec{A}$

Jaffe decomposition



light-cone framework & gauge $A^+ = 0$

$$\frac{1}{2} = \sum_q \frac{1}{2}\Delta q + \mathcal{L}_q + \Delta G + \mathcal{L}_g$$

$$\mathcal{L}_q = \int d^3r \langle P,S| \, \bar{q}(\vec{r})\gamma^+\left(\vec{r}\times i\vec{\partial}\right)^z q(\vec{r}) |P,S\rangle$$

$$\Delta G = \varepsilon^{+-ij}\int d^3r \, \langle P,S| \, \mathrm{Tr} F^{+i} A^j \, |P,S\rangle$$

$$\mathcal{L}_g = 2\int d^3r \langle P,S| \mathrm{Tr} F^{+j}\left(\vec{x}\times i\vec{\partial}\right)^z A^j |P,S\rangle$$

scalar diquark model

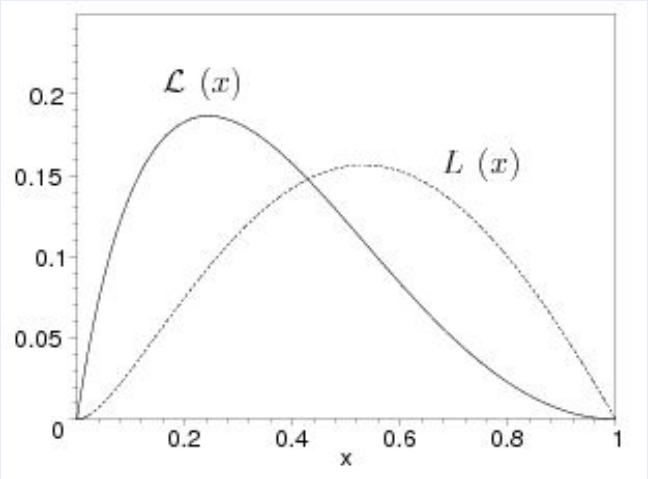
- LC wave functions $\psi_s^S(x, \mathbf{k}_\perp)$

$\hookrightarrow$ $\mathcal{L}_q$ from $|\psi_s^S(x, \mathbf{k}_\perp)|^2$

- GPDs from overlap integrals of $\psi^\dagger \psi$

$\hookrightarrow$ L_q from Ji

- $L_q = \mathcal{L}_q$.
  No surprise since
  $L_q - \mathcal{L}_q \sim \langle q^\dagger \vec{r} \times \vec{A} q \rangle$ and no $\vec{A}$ in scalar diquark model
- $L_q(x) \neq \mathcal{L}_q(x)$



scalar diquark model

- interpretation of
  $J_q(x) \equiv \frac{x}{2}\left[q(x) + E^q(x,0,0)\right]$
  <u>not</u> that of distribution of AM in x
- FT of
  $J(t) \equiv \frac{x}{2}\left[q(x) + E^q(x,0,0)\right]$
  <u>not</u> distribution of J_q^z in $\mathbf{b}_\perp$

M.B. + Hikmat BC,
PRD **79**, 071501 (2009)

QED for dressed e^- in QED

- LC wave functions $\psi_{sh}^S(x, \mathbf{k}_\perp)$

$\hookrightarrow$ $\mathcal{L}_q$ from $|\psi_{sh}^S(x, \mathbf{k}_\perp)|^2$

- GPDs from overlap integrals of $\psi^\dagger \psi$

$\hookrightarrow$ L_q from Ji

- $\mathcal{L}_e = L_e + \frac{\alpha}{4\pi} \neq L_e$

Ji decomposition

$$\frac{1}{2} = \sum_q \frac{1}{2}\Delta q + L_q + J_g$$

$$\frac{1}{2}\Delta q = \frac{1}{2}\int d^3x \, \langle P,S| \, q^\dagger(\vec{x})\Sigma^3 q(\vec{x}) \, |P,S\rangle$$

$$L_q = \int d^3x \langle P,S| q^\dagger(\vec{x}) \left(\vec{x} \times i\vec{D}\right)^3_z q(\vec{x}) |P,S\rangle$$

$$J_g = \int d^3x \, \langle P,S| \left[\vec{x} \times \left(\vec{E} \times \vec{B}\right)\right]^3 |P,S\rangle$$

- $i\vec{D} = i\vec{\partial} - g\vec{A}$

Jaffe decomposition

light-cone framework & gauge $A^+ = 0$

$$\frac{1}{2} = \sum_q \frac{1}{2}\Delta q + \mathcal{L}_q + \Delta G + \mathcal{L}_g$$

$$\mathcal{L}_q = \int d^3r \langle P,S| \, \bar{q}(\vec{r})\gamma^+ \left(\vec{r} \times i\vec{\partial}\right)^z_z q(\vec{r}) |P,S\rangle$$

$$\Delta G = \varepsilon^{+-ij} \int d^3r \, \langle P,S| \, \mathrm{Tr} F^{+i} A^j \, |P,S\rangle$$

$$\mathcal{L}_g = 2\int d^3r \langle P,S| \mathrm{Tr} F^{+j} \left(\vec{x} \times i\vec{\partial}\right)^z_z A^j |P,S\rangle$$

- GPDs $\longrightarrow L^q$
- $\overrightarrow{p}\overleftarrow{p} \longrightarrow \Delta G \longrightarrow \mathcal{L} \equiv \sum_{i \in q,g} \mathcal{L}^i$
- $L^q \neq \mathcal{L}^q$
- $\mathcal{L}^q - L^q = ?$
  - can we calculate/predict the difference?
  - what does it represent?

Wigner Functions (Belitsky, Ji, Yuan; Netz et al.)

$$W(x,\vec{b}_\perp,\vec{k}_\perp) \equiv \int \frac{d^2\vec{q}_\perp}{(2\pi)^2}\int \frac{d^2\xi_\perp d\xi^-}{(2\pi)^3} e^{ik\cdot\xi}\langle P'S'|\bar{q}(0)\gamma^+ q(\xi)|PS\rangle.$$

- (quasi) probabilty distribution for $\mathbf{b}_\perp$ and $\mathbf{k}_\perp$
- $f(x,\mathbf{k}_\perp) = \int d^2\mathbf{b}_\perp W(x,\vec{b}_\perp,\vec{k}_\perp)$
- $q(x,\mathbf{b}_\perp) = \int d^2\mathbf{k}_\perp W(x,\vec{b}_\perp,\vec{k}_\perp)$

OAM from Wigner (Lorcé et al.)

$$L_z = \int dx \int d^2\mathbf{b}_\perp \int d^2\mathbf{k}_\perp W(x,\vec{b}_\perp,\vec{k}_\perp)(b_x k_y - b_y k_x)$$
$$= \int d^3r \langle P,S|\,\bar{q}(\vec{r})\gamma^+\left(\vec{r}\times i\vec{\partial}\right)^z q(\vec{r})|P,S\rangle = \mathcal{L}^q$$

- same as Jaffe-Manohar definition

Gauge Invariance?

need to include Wilson-line gauge link to connect 0 and ξ

Wigner Functions with gauge link $\mathcal{U}_{0\xi}$ (Ji, Yuan; Hatta)

$$W(x,\vec{b}_\perp,\vec{k}_\perp) \equiv \int \frac{d^2\vec{q}_\perp}{(2\pi)^2}\int \frac{d^2\xi_\perp d\xi^-}{(2\pi)^3} e^{ik\cdot\xi}\langle P'S'|\bar{q}(0)\gamma^+\mathcal{U}_{0\xi}q(\xi)|PS\rangle.$$

W and thus $L_z = \int dx \int d^2\mathbf{b}_\perp \int d^2\mathbf{k}_\perp W(x,\vec{b}_\perp,\vec{k}_\perp)(b_x k_y - b_y k_x)$ may depend on choice of path!

straight line (Ji et al.)

straigth Wilson line from 0 to ξ yields

$L_q =$

$\int d^3x \langle P,S|q^\dagger(\vec{x})\left(\vec{x}\times i\vec{D}\right)^3 q(\vec{x})|P,S\rangle$

- $i\vec{D} = i\vec{\partial} - g\vec{A}$
- same as Ji-OAM
- $f(x,\mathbf{k}_\perp) = \int d^2\mathbf{b}_\perp W(x,\vec{b}_\perp,\vec{k}_\perp)$ not the TMDs relevant for SIDIS (missing FSI!)

Wigner Functions with gauge link $\mathcal{U}_{0\xi}$ (Ji, Yuan; Hatta)

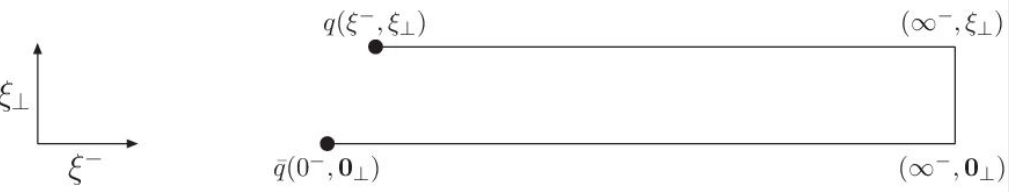
$$W(x,\vec{b}_\perp,\vec{k}_\perp) \equiv \int \frac{d^2\vec{q}_\perp}{(2\pi)^2}\int \frac{d^2\xi_\perp d\xi^-}{(2\pi)^3} e^{ik\cdot\xi}\langle P'S'|\bar{q}(0)\gamma^+\mathcal{U}_{0\xi}q(\xi)|PS\rangle.$$

W and thus $L_z = \int dx \int d^2\mathbf{b}_\perp \int d^2\mathbf{k}_\perp W(x,\vec{b}_\perp,\vec{k}_\perp)(b_x k_y - b_y k_x)$ may depend on choice of path!

Light-Cone Staple for $\mathcal{U}_{0\xi}^{\pm LC}$ (Hatta)

- want Wigner function that yields TMDs relevant for SIDIS when integrated $d^2\mathbf{b}_\perp$

$\hookrightarrow$ path for gauge link $\longrightarrow$ 'light-cone staple' $\longrightarrow \mathcal{U}_{0\xi}^{+LC}$



- Ji et al.: no link at $x^- = \infty$

$$\mathcal{L}_+^q = \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+\left(\vec{x}\times i\vec{\mathcal{D}}\right)^3 q(\vec{x})|P,S\rangle$$

- $i\vec{\mathcal{D}} = i\vec{\partial} - g\vec{A}(x^- = \infty, \mathbf{x}_\perp)$
- staple at $x^- = -\infty$ $\quad \mathcal{L}_-^q$
- PT $\Rightarrow$ $\quad \mathcal{L}_-^q = \mathcal{L}_+^q$
- $A_\perp(\infty,\mathbf{x}_\perp) = A_\perp(-\infty,\mathbf{x}_\perp) \Rightarrow \mathcal{L}_+^q = \mathcal{L}_{JM}^q$

$\hookrightarrow$ link at $x^- = \pm\infty$ no role for OAM!

straight line (→Ji)

$$L_q = \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+\left(\vec{x}\times i\vec{D}\right)^z q(\vec{x})|P,S\rangle$$

- $i\vec{D} = i\vec{\partial} - g\vec{A}$

light-cone staple (→ Jaffe-Manohar)

$$\mathcal{L}^q = \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+\left(\vec{x}\times i\vec{\mathcal{D}}\right)^z q(\vec{x})|P,S\rangle$$

- $i\vec{\mathcal{D}} = i\vec{\partial} - g\vec{A}(x^- = \infty, \mathbf{x}_\perp)$

difference $\mathcal{L}^q - L^q$

$$\mathcal{L}^q - L^q = -g \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+[\vec{x}\times(\mathbf{A}(\infty,\mathbf{x}_\perp) - \mathbf{A}(\vec{x}))]^z q(\vec{x})|P,S\rangle$$

$$\mathbf{A}_\perp(\infty,\mathbf{x}_\perp) - \mathbf{A}_\perp(\vec{x}) = \int_{x^-}^{\infty} dr^- F^{+\perp}(r^-,\mathbf{x}_\perp)$$

$$\sqrt{2}F^{+y} = F^{0y} + F^{zy} = -E^y + B^x$$

Difference Jaffe/Manohar vs. Ji

straight line (→Ji)

$$L_q = \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+\left(\vec{x}\times i\vec{D}\right)^z q(\vec{x})|P,S\rangle$$

- $i\vec{D} = i\vec{\partial} - g\vec{A}$

light-cone staple (→ Jaffe-Manohar)

$$\mathcal{L}^q = \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+\left(\vec{x}\times i\vec{\mathcal{D}}\right)^z q(\vec{x})|P,S\rangle$$

- $i\vec{\mathcal{D}} = i\vec{\partial} - g\vec{A}(x^- = \infty, \mathbf{x}_\perp)$

difference $\mathcal{L}^q - L^q$

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$$\mathbf{A}_\perp(\infty,\mathbf{x}_\perp) - \mathbf{A}_\perp(\vec{x}) = \int_{x^-}^{\infty} dr^- F^{+\perp}(r^-,\mathbf{x}_\perp)$$

color Lorentz Force acting on ejected quark (MB: arXiv:08103589)

$$\sqrt{2}F^{+y} = F^{0y} + F^{zy} = -E^y + B^x = -\left(\vec{E} + \vec{v}\times\vec{B}\right)^y \text{ for } \vec{v} = (0,0,-1)$$

Difference Jaffe/Manohar vs. Ji

straight line (→Ji)

$$L_q = \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+\left(\vec{x}\times i\vec{D}\right)^z q(\vec{x})|P,S\rangle$$

- $i\vec{D} = i\vec{\partial} - g\vec{A}$

light-cone staple (→ Jaffe-Manohar)

$$\mathcal{L}^q = \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+\left(\vec{x}\times i\vec{\mathcal{D}}\right)^z q(\vec{x})|P,S\rangle$$

- $i\vec{\mathcal{D}} = i\vec{\partial} - g\vec{A}(x^- = \infty, \mathbf{x}_\perp)$

difference $\mathcal{L}^q - L^q$

$$\mathcal{L}^q - L^q = -g \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+[\vec{x}\times(\mathbf{A}(\infty,\mathbf{x}_\perp) - \mathbf{A}(\vec{x}))]^z q(\vec{x})|P,S\rangle$$

$$\mathbf{A}_\perp(\infty,\mathbf{x}_\perp) - \mathbf{A}_\perp(\vec{x}) = \int_{x^-}^{\infty} dr^- F^{+\perp}(r^-,\mathbf{x}_\perp)$$

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Torque along the trajectory of q

$$T^z = \left[\vec{x}\times\left(\vec{E} - \hat{\vec{z}}\times\vec{B}\right)\right]^z$$

Change in OAM

$$\Delta L^z = \int_{x^-}^{\infty} dr^- \left[\vec{x}\times\left(\vec{E} - \hat{\vec{z}}\times\vec{B}\right)\right]^z$$

straight line (→Ji)

$$L_q = \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+\left(\vec{x}\times i\vec{D}\right)^z q(\vec{x})|P,S\rangle$$

- $i\vec{D} = i\vec{\partial} - g\vec{A}(\vec{x})$

light-cone staple (→ Jaffe-Manohar)

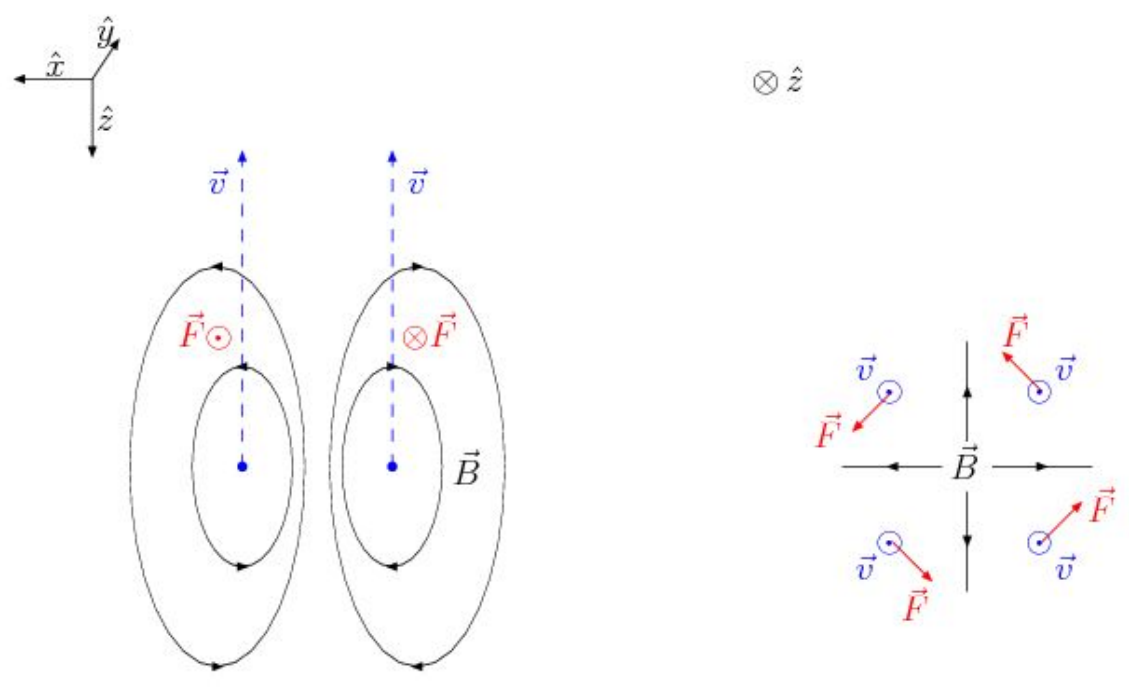
$$\mathcal{L}^q = \int d^3x \langle P,S|\bar{q}(\vec{x})\gamma^+\left(\vec{x}\times i\vec{\mathcal{D}}\right)^z q(\vec{x})|P,S\rangle$$

- $i\vec{\mathcal{D}} = i\vec{\partial} - g\vec{A}(x^- = \infty, \mathbf{x}_\perp)$

difference $\mathcal{L}^q - L^q$

$\mathcal{L}^q - L^q$ = change in OAM as the quark leaves the nucleon

example: torque in magnetic dipole field



- $E^q(x, 0, -\mathbf{\Delta}_\perp^2) \longrightarrow \perp$ deformation of PDFs for $\perp$ polarized target

$\hookrightarrow$ parton interpretation for Ji-relation

- (in principle) $GPD(x, \xi)$ from QCD evolution of $GPD(\xi, \xi, Q^2)$
- interpretation of $L_q - \mathcal{L}_q$ as change in OAM of ejected quark

- L_q matrix element of

$$q^\dagger \left[\vec{r} \times \left(i\vec{\partial} - g\vec{A}\right)\right]^z q = \bar{q}\gamma^0 \left[\vec{r} \times \left(i\vec{\partial} - g\vec{A}\right)\right]^z q$$

- $\mathcal{L}_q^z$ matrix element of $(\gamma^+ = \gamma^0 + \gamma^z)$

$$\bar{q}\gamma^+ \left[\vec{r} \times i\vec{\partial}\right]^z q\Big|_{A^+=0}$$

- (for $\vec{p} = 0$) matrix element of $\bar{q}\gamma^z \left[\vec{r} \times \left(i\vec{\partial} - g\vec{A}\right)\right]^z q$ vanishes (parity!)

$\hookrightarrow$ L_q identical to matrix element of $\bar{q}\gamma^+ \left[\vec{r} \times \left(i\vec{\partial} - g\vec{A}\right)\right]^z q$ (nucleon at rest)

$\hookrightarrow$ even in light-cone gauge, L_q^z and $\mathcal{L}_q^z$ still differ by matrix element of $q^\dagger \left(\vec{r} \times g\vec{A}\right)^z q\Big|_{A^+=0} = q^\dagger \left(r^x g A^y - r^y g A^x\right) q\Big|_{A^+=0}$

- how significant is that difference?

first: QED without electrons

- apply $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{b}(\vec{a} \cdot \vec{c})$ to $\vec{E} \times \left(\vec{\nabla} \times \vec{A}\right)$

$$\begin{aligned} \vec{J} &= \int d^3r\, \vec{x} \times \left(\vec{E} \times \vec{B}\right) = \int d^3r\, \vec{x} \times \left[\vec{E} \times \left(\vec{\nabla} \times \vec{A}\right)\right] \\ &= \int d^3r \left[E^j \left(\vec{x} \times \vec{\nabla}\right) A^j - \vec{x} \times (\vec{E} \cdot \vec{\nabla})\vec{A}\right] \end{aligned}$$

- integrate by parts (drop surface term)

$$\vec{J} = \int d^3r \left[E^j \left(\vec{x} \times \vec{\nabla}\right) A^j + \left(\vec{x} \times \vec{A}\right) \vec{\nabla} \cdot \vec{E} + \vec{E} \times \vec{A}\right]$$

- drop 2nd term (eq. of motion $\vec{\nabla} \cdot \vec{E} = 0$), yielding $\vec{J} = \vec{L} + \vec{S}$ with

$$\vec{L} = \int d^3r\, E^j \left(\vec{x} \times \vec{\nabla}\right) A^j \qquad \vec{S} = \int d^3r\, \vec{E} \times \vec{A}$$

- note: $\vec{L}$ and $\vec{S}$ not separately gauge invariant as written, but can be made so ($\rightarrow$ nonlocal)

QED with electrons

$$\begin{aligned}\vec{J}_\gamma &= \int d^3r\, \vec{r} \times \left(\vec{E} \times \vec{B}\right) = \int d^3r\, \vec{r} \times \left[\vec{E} \times \left(\vec{\nabla} \times \vec{A}\right)\right] \\ &= \int d^3r \left[E^j \left(\vec{r} \times \vec{\nabla}\right) A^j - \vec{r} \times (\vec{E} \cdot \vec{\nabla})\vec{A}\right] \\ &= \int d^3r \left[E^j \left(\vec{r} \times \vec{\nabla}\right) A^j + \left(\vec{r} \times \vec{A}\right) \vec{\nabla} \cdot \vec{E} + \vec{E} \times \vec{A}\right]\end{aligned}$$

- replace 2nd term (eq. of motion $\vec{\nabla} \cdot \vec{E} = ej^0 = e\psi^\dagger\psi$), yielding

$$\vec{J}_\gamma = \int d^3r \left[\psi^\dagger \vec{r} \times e\vec{A}\psi + E^j \left(\vec{x} \times \vec{\nabla}\right) A^j + \vec{E} \times \vec{A}\right]$$

- $\psi^\dagger \vec{r} \times e\vec{A}\psi$ cancels similar term in electron OAM $\psi^\dagger \vec{r} \times (\vec{p} - e\vec{A})\psi$

$\hookrightarrow$ decomposing $\vec{J}_\gamma$ into spin and orbital also shuffles angular momentum from photons to electrons!

- can also be done for only part of $\vec{A} \rightarrow$ Chen/Goldman, Wakamatsu